

# From symmetry-protected topological order to Landau order

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## Abstract

Focusing on the particular case of the discrete symmetry group  $\mathbb{Z}_N \times \mathbb{Z}_N$ , we establish a mapping between symmetry protected topological phases and symmetry broken phases for one-dimensional spin systems. It is realized in terms of a non-local unitary transformation which preserves the locality of the Hamiltonian. We derive the image of the mapping for various phases involved, including those with a mixture of symmetry breaking and topological protection. Our analysis also applies to topological phases in spin systems with arbitrary continuous symmetries of unitary, orthogonal and symplectic type. This is achieved by identifying suitable subgroups  $\mathbb{Z}_N \times \mathbb{Z}_N$  in all these groups, together with a bijection between the individual classes of projective representations.

## 1 Introduction

Symmetry protected topological phases received a lot of interest recently due to their characteristic properties such as the existence of massless boundary modes. Prominent examples are various types of topological insulators [1, 2] and spin systems such as the AKLT model [3, 4]. In contrast to purely topological phases such as the fractional or integer quantum Hall effect, in these systems the robustness of boundary modes is directly tied to the presence of symmetries.

For one-dimensional spin systems a complete classification of gapped symmetry protected topological and symmetry broken phases has been established in Refs. [5, 6, 7, 8]. Restricting one's attention to on-site symmetries  $G$  only, the phases are fully characterized by the spontaneous symmetry breaking of  $G$  to a subgroup  $K$ , together with an element from the cohomology group  $H^2(K, U(1))$ . The latter labels the distinct classes of projective representations of  $K$  and can be thought of as being a discrete topological invariant attached to edge modes of the system.<sup>1</sup>

In the present paper we shall focus on the particular symmetry group  $\mathbb{Z}_N \times \mathbb{Z}_N$ . It is the smallest abelian group which exhibits up to  $N$  distinct topological phases. At the same time, it allows to study topological phases in combination with the phenomenon of spontaneous symmetry breaking if  $N$  has non-trivial divisors. While information about the latter can be inferred from suitable local Landau order parameters, the detection of topological phases in 1D systems requires the use of non-local string order parameters (see, e.g., [9, 10, 11, 12, 13, 14]). As we shall see later, discrete groups of type  $\mathbb{Z}_N \times \mathbb{Z}_N$  also play a distinguished role when extending our considerations to continuous symmetry groups.

Before we proceed, let us briefly review the specific case of the dihedral group  $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$  which historically arose in connection with the  $SO(3)$  AKLT model [3, 4]. It is well-known that the AKLT model realizes the Haldane phase of  $S = 1$  spin models. It exhibits topological order which can be detected using the non-local string order parameter suggested by Rommelse and Den Nijs [9]. Soon

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<sup>1</sup>In the case of periodic boundary conditions, these edge modes arise virtually in the groundstate entanglement spectrum after the system is cut into two pieces.

after, it was discovered that the presence of topological order could be interpreted as the spontaneous breaking of a “hidden”  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetry which is related to the occurrence of two spin 1/2 degrees of freedom at the edges of the chain [15]. This symmetry breaking becomes manifest after a non-local unitary transformation which maps the original string order parameter to a local Landau order parameter. It is known that this observation extends to AKLT chains based on higher integer spins  $S$  [16]. However, it only became clear later that the topological protection does in fact not rely on the full  $SO(3)$  symmetry but that it can already be achieved by restricting one’s attention to a discrete subgroup  $\mathbb{Z}_2 \times \mathbb{Z}_2$  [17, 18]. The non-trivial elements of this group can be thought of as rotations by  $\pi$  around three mutually orthogonal axes.

In this paper, we generalize the previous ideas to arbitrary groups  $\mathbb{Z}_N \times \mathbb{Z}_N$ . In the process we face two difficulties: First of all, these groups allow for more complicated patterns of spontaneous symmetry breaking, giving rise to a whole hierarchy of phases. And secondly, a more refined version of string order parameter has to be used. Indeed, while for  $\mathbb{Z}_2 \times \mathbb{Z}_2$  one only has to distinguish between two topological phases (trivial and non-trivial), we now have to discriminate  $N$  distinct phases which are labeled by a parameter  $t \in \mathbb{Z}_N$ . It should be obvious that such a number cannot be extracted from a single expectation value since there is no reason why the latter should be quantized.

As it turns out, the method of choice is to employ the selection rule procedure of Ref. [13].<sup>2</sup> Our analysis starts with a specific (string) order parameter  $S(a, b)$  which depends on two parameters  $a, b \in \mathbb{Z}_N$  and which vanishes except if the selection rule  $a + tb = 0$  modulo  $N$  is satisfied. While  $S(a, b)$  is non-local for  $b \neq 0$  it becomes local for  $b = 0$ . Determining  $S(a, b)$  for various choices of  $a$  and  $b$ , one can thus extract information about the topological phase  $t$  and about the potential existence of spontaneous symmetry breaking. We then construct a non-local unitary transformation  $U_N$  which maps  $S(a, b)$  to  $S(a, a + b)$ . We analyze the implications of this mapping and find that purely topological phases are mapped to symmetry breaking ones. The transformation  $U_N$  thus allows to re-interpret topological order in terms of standard local Landau order. More generally, we work out the effect of acting with  $U_N$  on almost all phases of  $\mathbb{Z}_N \times \mathbb{Z}_N$  spin chains, including those with a mixture of topological protection and spontaneous symmetry breaking to a subgroup  $\mathbb{Z}_r \times \mathbb{Z}_r$ .<sup>3</sup>

As the original example of the  $SO(3)$  AKLT model suggests, the results we derive may equally well be applied to the detection of topological order in systems with continuous symmetry groups. This is due to the fact that each continuous symmetry group which permits non-trivial topological phases (see Ref. [8]), contains a non-trivial subgroup of the form  $\mathbb{Z}_N \times \mathbb{Z}_N$  for a suitable choice of  $N \geq 2$ . Moreover, the projective representations characterizing the topological phases with regard to either the continuous group or its discrete subgroup are in bijection.<sup>4</sup> For all groups of unitary, orthogonal and symplectic type the relevant subgroups are constructed explicitly in Sect. 5.

The paper is structured as follows. In Sect. 2 we introduce the group  $\mathbb{Z}_N \times \mathbb{Z}_N$  and we discuss its projective representations. Furthermore, we introduce the (string) order parameter  $S(a, b)$  and explain how it can be used to characterize the distinct topological and symmetry broken phases of  $\mathbb{Z}_N \times \mathbb{Z}_N$ -invariant spin systems. The construction of the non-local unitary transformation  $U_N$  which maps purely topological phases to symmetry broken ones is the content of Sect. 3. In Sect. 4 we analyze the fate of each individual phase under the action of  $U_N$ . Finally, Sect. 5 discusses the implications of our results for continuous groups. We conclude with a brief summary and an outlook to future directions.

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<sup>2</sup>Other discriminating string order parameters with a different scope of applicability have been suggested in [12, 14].

<sup>3</sup>These subgroups are those which are relevant for understanding the topological phases of systems with continuous symmetry.

<sup>4</sup>This last statement is not valid for the series  $PSO(4N)$ . Here, one needs to work with a discrete abelian subgroup involving more than two factors.

## 2 Preliminaries

Different phases of one-dimensional quantum systems can either arise due to symmetry breaking, due to topology, or a combination thereof. We will consider systems with an on-site symmetry  $G$ , but we will not impose space-time symmetries such as translation invariance, time reversal or inversion symmetry. If the groundstate of the system breaks the symmetry to a subgroup  $K \subset G$ , then the possible topological phases are given by different classes of projective representations of  $K$  [6, 7].

The group  $\mathbb{Z}_N \times \mathbb{Z}_N$  has  $N$  different projective classes. Let  $R$  and  $\tilde{R}$  be the generators of this group and let  $R'$  and  $\tilde{R}'$  be projective representations of these generators. The phases  $R'^N = e^{i\theta}$  and  $\tilde{R}'^N = e^{i\tilde{\theta}}$  can be removed by a redefinition (gauge transformation) of  $R'$  and  $\tilde{R}'$ . However, the phase  $R'\tilde{R}'R'^{-1}\tilde{R}'^{-1} = e^{i\phi}$  is gauge invariant and determines the projective class of the corresponding representation. Moreover,  $\phi$  is an integer multiple of  $\frac{2\pi}{N}$  due to the cyclic property of the group  $\mathbb{Z}_N$ . Thus the projective class  $t \in \mathbb{Z}_N$  of a representation of  $\mathbb{Z}_N \times \mathbb{Z}_N$  can be obtained from the relation

$$R'\tilde{R}' = \omega^t \tilde{R}'R' \quad . \quad (2.1)$$

Here, we used the abbreviation  $\omega = \exp(2\pi i/N)$ .

The special case of  $N = 2$  arises in  $S = 1$  spin chains in which  $\mathbb{Z}_2 \times \mathbb{Z}_2$  is a subgroup of  $SU(2)$  generated by  $R^x = \exp(i\pi S^x)$  and  $R^z = \exp(i\pi S^z)$ . In these systems two phases can occur: a topological trivial and a non-trivial (Haldane) phase. The latter is characterized by a hidden symmetry breaking which comes manifest after applying a non-local unitary transformation (NL-UT) [15]. The NL-UT can be written as [16]:

$$U_2 = \prod_{i < j} \exp(\pi i S_i^x S_j^z) \quad . \quad (2.2)$$

This transformation preserves the symmetry and maps local invariant Hamiltonians (such as that of the XYZ or the bi-linear bi-quadratic model) to local Hamiltonians. Most importantly, it maps the string order parameter  $S_i^a \prod_{i \leq k < j} R_k^a S_j^a$  to a Landau order parameter  $S_i^a S_j^a$  for  $a = x$  or  $z$ , which explains that string order and hidden symmetry breaking are one-to-one related to each other.<sup>5</sup>

Our attempt to generalize the previous considerations to  $\mathbb{Z}_N \times \mathbb{Z}_N$  heavily used the string order selection rules introduced in Ref. [13]. In order to explain the underlying ideas, let us consider a spin chain which is invariant under two commuting transformations  $\mathcal{X}$  and  $\mathcal{Y}$ . Denote the restrictions of these transformations to the right boundary modes by  $\mathcal{X}'$  and  $\mathcal{Y}'$ . Since the boundary modes only need to transform projectively, we have that  $\mathcal{X}'\mathcal{Y}' = e^{i\phi}\mathcal{Y}'\mathcal{X}'$  where the phase  $\phi$  determines the projective class of the representation of the right boundary mode. The phase  $\phi$  thus also determines the topological phase of the system. Let us now consider a string order parameter

$$O_i^L \prod_{i \leq k < j} \mathcal{X}_k O_j^R \quad . \quad (2.3)$$

A non-vanishing expectation value of such a string order parameter in the limit  $|i - j| \rightarrow \infty$  implies invariance under the transformation  $\mathcal{X}$ , but stated as such it contains no information on the topological phase [10]. However, the latter can be gained from a group theoretical selection rule [13]. If the operators  $O^L$  and  $(O^R)^\dagger$  have the same quantum number with respect to  $\mathcal{Y}$ ,

$$\mathcal{Y}^{-1}O^L\mathcal{Y} = e^{i\sigma}O^L \quad \text{and} \quad \mathcal{Y}^{-1}O^R\mathcal{Y} = e^{-i\sigma}O^R \quad , \quad (2.4)$$

then the selection rule states that the string order parameter can only be nonzero if  $\sigma = \phi$ . For the case of systems with symmetry  $\mathbb{Z}_N \times \mathbb{Z}_N$  the role of  $\mathcal{X}$  and  $\mathcal{Y}$  are played by  $R$  and  $\tilde{R}$ , respectively.

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<sup>5</sup>Here and in what follows we deviate from the standard notation by letting the string extend to site  $i$ . This has the benefit of simplifying subsequent formulas.

We define operators  $X^a$  which are invariant under  $R$  but which have a specific quantum number with respect to  $\tilde{R}$ :

$$\tilde{R}^{-1} X^a \tilde{R} = \omega^{-a} X^a \quad , \quad [R, X^a] = 0 \quad . \quad (2.5)$$

Using these operators, we introduce the string order parameter

$$S(a, b) = X_i^a \prod_{i \leq k < j} R_k^b (X_j^a)^{-1} \quad . \quad (2.6)$$

We note that this operator becomes local for  $b = 0$ . The selection rule for string order states that the expectation value

$$\Sigma(a, b) = \lim_{|i-j| \rightarrow \infty} \langle S(a, b) \rangle \quad (2.7)$$

can only be nonzero if  $a + tb = 0$  modulo  $N$ , where  $t$  is the projective class of the right boundary mode [13], see Eq. (2.1).

### 3 A non-local unitary transformation for $\mathbb{Z}_N \times \mathbb{Z}_N$

In this section we aim to generalize the NL-UT given in Eq. (2.2), such that it is applicable to systems with  $\mathbb{Z}_N \times \mathbb{Z}_N$  symmetry. Recall that this group is generated by the symmetries  $R$  and  $\tilde{R}$ . Furthermore we consider two operators  $O$  and  $\tilde{O}$  which have the properties  $R = \omega^O$  and  $\tilde{R} = \omega^{\tilde{O}}$ , with  $\omega = \exp(2\pi i/N)$  as in the previous section. These operators will generalize  $S^x$  and  $S^z$ . We then define a NL-UT as

$$U_N = \prod_{i < j} \omega^{O_i \tilde{O}_j} \quad . \quad (3.1)$$

All terms in the above product commute with each other. Commutators which are possibly nonzero are of the form  $[\omega^{O_i \tilde{O}_j}, \omega^{O_j \tilde{O}_k}]$ . They can be rewritten as  $[\tilde{R}_j^{O_i}, R_j^{\tilde{O}_k}]$ , from which it is clear that also these commutators are zero since  $R$  and  $\tilde{R}$  commute. Using similar arguments it is easily shown that both  $R$  and  $\tilde{R}$  commute with  $U_N$ . Thus  $U_N$  preserves the  $\mathbb{Z}_N \times \mathbb{Z}_N$  symmetry generated by these two transformations.

Consider a  $\mathbb{Z}_N \times \mathbb{Z}_N$  invariant local Hamiltonian  $H_0$ . We will now show that the transformed Hamiltonian  $H_1 = U_N^{-1} H_0 U_N$  is also local. More precisely,  $n$ -body terms (which act on  $n$  consecutive sites) will be mapped to  $n$ -body terms. We will show this for  $n = 2$ . Let  $h_{i,i+1}$  be a term acting on sites  $i$  and  $i + 1$ . This term is transformed as follows:

$$U_N^{-1} h_{i,i+1} U_N = \prod_{j < i} \tilde{R}_{i,i+1}^{-O_j} \prod_{j > i+1} R_{i,i+1}^{-\tilde{O}_j} \omega^{-O_i \tilde{O}_{i+1}} h_{i,i+1} \omega^{O_i \tilde{O}_{i+1}} \prod_{j > i+1} R_{i,i+1}^{\tilde{O}_j} \prod_{j < i} \tilde{R}_{i,i+1}^{O_j} \quad (3.2)$$

$$= \omega^{-O_i \tilde{O}_{i+1}} h_{i,i+1} \omega^{O_i \tilde{O}_{i+1}} \quad . \quad (3.3)$$

Here  $R_{i,i+1}$  is short for  $R_i R_{i+1}$ . The simplification is due to the  $\mathbb{Z}_N \times \mathbb{Z}_N$  invariance of the Hamiltonian. The result is clearly a local 2-body term. The generalization to  $n$ -body Hamiltonians is straightforward.

In the previous section it was explained that the string order parameter  $S(a, b)$  is able to detect topological order via the selection rule. We will now discuss the transformation rule of  $S(a, b)$ . The operators  $X^a$  appearing in  $S(a, b)$  transform as

$$U_N^{-1} X_i^a U_N = \prod_{j < i} \tilde{R}_i^{-O_j} X_i^a \prod_{j < i} \tilde{R}_i^{O_j} = \prod_{j < i} \omega^{-a O_j} X_i^a \quad . \quad (3.4)$$

In these equalities we have used Eq. (2.5), in particular that  $X^a$  and  $R$  commute. With this transformation rule it follows that:

$$U_N^{-1}S(a,b)U_N = U_N^{-1}X_i^a \prod_{i \leq k < j} \omega^{bO_k} X_j^{-a} U_N = X_i^a \prod_{i \leq k < j} \omega^{(a+b)O_k} X_j^{-a} = S(a, a+b) \quad . \quad (3.5)$$

Applying  $U_N$  sufficiently many times ( $n$  times, such that  $b + na = 0$  modulo  $N$ ), the result will eventually be  $U_N^{-n}S(a,b)U_N^n = S(a,0) = X_i^a X_j^{-a}$ . The operator  $U_N^n$  thus relates the string order parameter which is capable of detecting topological phases to a Landau order parameter measuring symmetry breaking. Indeed, a nonzero  $S(a,b)$  gives information on the topological phase through the selection rule, whereas a nonzero  $S(a,0)$  gives information on the breaking of the symmetry generated by  $\tilde{R}$ .

Just as before, we can define the string order parameter  $\tilde{S}^{(a,b)} = \tilde{X}_i^a \prod_{i \leq k < j} \tilde{R}_k^b \tilde{X}_j^{-a}$  with operators  $\tilde{X}^a$  satisfying  $R^{-1}\tilde{X}^a R = \omega^a \tilde{X}^a$  and  $[\tilde{X}^a, \tilde{R}] = 0$ . Similar to Eq. (3.5) we have the transformation rule  $U_N^{-1}\tilde{S}^{(a,b)}U_N = \tilde{S}^{(a,a+b)}$ . In the topological phase labeled by  $t$  both  $S^{(a,b)}$  and  $\tilde{S}^{(a,b)}$  can be nonzero (when their arguments satisfy  $a + tb = 0 \pmod{N}$ ). If both string order parameters are zero, the unbroken symmetry transformations of the  $U_N$ -transformed system form a group of the form  $\mathbb{Z}_r \times \mathbb{Z}_r \subset \mathbb{Z}_N \times \mathbb{Z}_N$ .

## 4 A mapping of phases

In this section we will discuss in detail what will happen to the symmetries after performing the NL-UT. That is, we start with a system with symmetry  $\mathbb{Z}_N \times \mathbb{Z}_N$  whose ground states spontaneously break the symmetry to  $\mathbb{Z}_{r_0} \times \mathbb{Z}_{r_0}$ , with  $q_0 r_0 = N$ . We furthermore assume that the system is in the topological phase  $t_0$ , defined by the projective class of the right boundary modes. In the previous section it was argued that the transformed system, obtained by applying the NL-UT defined in Eq. (3.1), could show a different pattern of symmetry breaking and could reside in a different topological phase. It was also argued that the group of unbroken symmetries of the transformed system is of the form  $\mathbb{Z}_{r_1} \times \mathbb{Z}_{r_1}$ , with  $q_1 r_1 = N$ . Let  $t_1$  label the topological phase of the transformed system. We aim to find the explicit form of the relation

$$f_N : (r_0, t_0) \xrightarrow{U_N} (r_1, t_1) \quad . \quad (4.1)$$

As a warmup we will first assume that  $N$  is prime. In this case, there is either no symmetry breaking or symmetry is fully broken. From the selection rule we conclude that only the string order parameter of the form  $S(at_0, -a)$  can be nonzero. This string order parameter is mapped to  $U_N^{-1}S(at_0, -a)U_N = S(at_0, a(t_0 - 1))$ . As long as  $t_0 \neq 1$  no symmetry is broken ( $r_1 = N$ ). The topological phase can be deduced from the selection rule:  $t_0 + t_1(t_0 - 1) = 0 \pmod{N}$ . In the exceptional case of  $t_0 = 1$ , the operator  $S(at_0, -a)$  is mapped to a Landau order parameter measuring symmetry breaking ( $r_1 = 1$  and trivially  $t_1 = 0$ ). Conversely, if we start with a symmetry breaking phase, ( $S(a, 0) \neq 0$ ), the transformed system will have nonzero string order parameter  $S(a, a)$  from which it follows that  $t_1 = N - 1$ . Note that the trivial phase is always mapped to the trivial phase. In Table 1 the action of the map  $f_N$  is illustrated for  $N = 5$ .

The discussion is slightly more involved when  $N$  is not prime because topological order can then be mixed with symmetry breaking order. The first step in the analysis is to use the selection rule to determine when  $S(a, b)$  is possibly nonzero. Note that  $b$  is always a multiple of  $q_0$  ( $b = n_1 q_0$ ). The transformations  $R^b$  and  $\tilde{R}^q$  restricted to the right boundaries do not commute but give rise to a phase factor. This complex phase depends on the topological phase  $t_0$  via Eq. (2.1):

$$\exp(2\pi i / r_0)^{t_0 n_1} = \omega^{t_0 b} \quad . \quad (4.2)$$

Moreover, from Eq. (2.5) it follows that transforming  $X^a$  by  $\tilde{R}^{q_0}$  gives rise to the phase  $\omega^{-aq_0}$ . The selection rule states that the string order parameter  $S(a, b)$  is nonzero only if these two phases coincide, thus if  $t_0 b + q_0 a = 0 \bmod N$ . We conclude that nonzero string order parameters are of the form  $S(n_1 t_0 + n_2 r_0, -n_1 q_0)$ . An extra term  $n_2 r_0$  in the first argument is allowed since  $X^{r_0}$  commutes with  $\tilde{R}^{q_0}$ . Setting  $n_1 = 0$  results in a nonzero Landau order parameter  $S(n_2 r_0, 0)$ , which is consistent with symmetry breaking at hand. This string order parameter is mapped to

$$S(n_1 t_0 + n_2 r_0, -n_1 q_0) \xrightarrow{U_N} S(n_1 t_0 + n_2 r_0, n_1(t_0 - q_0) + n_2 r_0) . \quad (4.3)$$

The transformed string order parameter is a Landau order parameter if its second argument vanishes (modulo  $N$ ). This can only happen if  $n_1(t_0 - q_0) = 0 \bmod r_0$ . The smallest  $n_1$  which fulfills this equation is given by  $n_1 = r_0 / \gcd(t_0 - q_0, r_0)$ . The corresponding symmetry breaking operator  $X^a$  is determined by  $a = n_1 t_0 + n_2 r_0 = n_1 q_0 = N / \gcd(t_0 - q_0, r_0)$ . We conclude that the symmetry of the transformed system is determined by

$$r_1 = \frac{N}{\gcd(t_0 - q_0, r_0)} . \quad (4.4)$$

Note that the second argument of the transformed string operator is a multiple of  $q_1 = \gcd(t_0 - q_0, r_0)$ . Thus the selection rule can be used to determine the topological phase  $t_1$  of the transformed system. From this rule we have that

$$(n_1 t_0 + n_2 r_0) q_1 = -t_1 [n_1(t_0 - q_0) + n_2 r_0] . \quad (4.5)$$

The solution for  $t_1$  should be independent of  $n_1$  and  $n_2$ . Thus factoring these constants out we obtain two equations

$$0 = t_1(t_0 - q_0) + t_0 q_1 \bmod N , \quad (4.6)$$

$$0 = r_0(q_1 + t_1) \bmod N . \quad (4.7)$$

From the second equation we deduce that  $t_1$  is equal to  $-q_1$  modulo  $q_0$  thus  $t_1 = nq_0 - q_1$ . Substituting in the first and simplifying results in  $n(t_0 - q_0) = -q_1 \bmod r_0$ . This equation has a solution for  $n$  due to the definition of  $q_1 = \gcd(t_0 - q_0, r_0)$ . In Table 1 the map  $f_N$  is worked out for  $N = 6$ .

Let us finally comment on a subtle technical issue. Our derivation of the map  $f_N$  hinges on the presence of (string) order via the selection rule. However, the selection rule only gives a necessary but not a sufficient condition on the non-vanishing of the (string) order. It can be accidentally absent at specific points in the phase diagram where one would have expected it to occur from the selection rule. To resolve this problem, we recall that the map  $f_N$  is not a statement about specific points in a phase but rather about phases as a whole. Assuming the existence of some point in the phase diagram where the groundstate leads to a non-vanishing string order is enough to show that the map  $f_N$  is valid for the whole phase.

Note that from Table 1 it can be seen that  $f_N$  is bijective for  $N = 5$  and  $N = 6$ . This is as expected since the spectrum and thus phase transitions should be invariant under the NL-UT. Also it has been checked numerically for values of  $N$  up to 100 that indeed  $f_N$  is bijective.

In Ref. [21] a different NL-UT transformation, mapping topological phases to symmetry breaking phases, is discussed. The main difference is that we discuss one single NL-UT whereas in [21] the NL-UT  $\mathcal{D}_t$  depends on the topological phase  $t$  at hand. The map  $\mathcal{D}_t$  always maps a pure topological phase to the phase where the full symmetry is spontaneously broken (SSB-phase), whereas  $U_N$  maps phases characterized by a mixture of symmetry breaking and topology to each other. Under certain conditions one can map such phases to the SSB-phase applying the transformation  $U_N$  sufficiently many times. Indeed, from Eq. (4.3) we have that

$$S(n_1 t_0 + n_2 r_0, -n_1 q_0) \xrightarrow{U_N^u} S(n_1 t_0 + n_2 r_0, n_1(ut_0 - q_0) + n_2 ur_0) , \quad (4.8)$$

$\mathbb{Z}_5 \times \mathbb{Z}_5$			$\mathbb{Z}_6 \times \mathbb{Z}_6$		
$t_0$		$t_1$	$(r_0, t_0)$		$(r_1, t_1)$
0	$\rightarrow$	0	(6, 0)	$\rightarrow$	(6, 0)
1	$\rightarrow$	SSB	(6, 1)	$\rightarrow$	(1, 0)SSB
2	$\rightarrow$	3	(6, 2)	$\rightarrow$	(6, 4)
3	$\rightarrow$	1	(6, 3)	$\rightarrow$	(3, 0)
4	$\rightarrow$	2	(6, 4)	$\rightarrow$	(2, 0)
SSB	$\rightarrow$	4	(6, 5)	$\rightarrow$	(3, 2)
			--- (3, 0) ---	$\rightarrow$	(6, 3)
			(3, 1)	$\rightarrow$	(6, 1)
			--- (3, 2) ---	$\rightarrow$	(2, 1)
			--- (2, 0) ---	$\rightarrow$	(6, 2)
			(2, 1)	$\rightarrow$	(3, 1)
			--- (1, 0)SSB ---	$\rightarrow$	(6, 5)

Table 1: Mapping of symmetry breaking and topological phases of a  $\mathbb{Z}_5 \times \mathbb{Z}_5$  (left) and a  $\mathbb{Z}_6 \times \mathbb{Z}_6$  (right) invariant system. With SSB we refer to the phase characterized by full spontaneous symmetry breaking.

where  $u$  is the number of times  $U_N$  is applied. An arbitrary phase is mapped to the SSB-phase if its string order parameters are mapped to Landau order parameters, that is if  $n_1(ut_0 - q_0) + n_2ur_0 = 0 \pmod N$  for all values of  $n_1$  and  $n_2$ . From this we have the conditions that  $ut_0 = q_0 \pmod N$  and  $u = 0 \pmod q_0$ . The first condition can only be satisfied if  $\gcd(t_0, r_0) = 1$ . This is equivalent to stating that the projective representations of  $\mathbb{Z}_{r_0} \times \mathbb{Z}_{r_0}$  in class  $t_0$  are *maximally non-commutative* [21], a requirement for defining the map  $\mathcal{D}_{t_0}$ . The second condition can be verified by considering orbits of  $f_N$  containing the SSB-phase. For example, taking  $N = 6$  this orbit is

$$\text{SSB} \rightarrow (6, 5) \rightarrow (3, 2) \rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (6, 1) \rightarrow \text{SSB} . \quad (4.9)$$

## 5 Relevance to systems with continuous symmetry groups

Although the previous section gives a better understanding of the connection between symmetry breaking order and topological order, as it stands it is only applicable to systems with the specific symmetry  $\mathbb{Z}_N \times \mathbb{Z}_N$ . In this section, however, we will argue that almost all connected and compact simple Lie groups  $G_\Gamma$  (with only one exception) contain a subgroup of the form  $\mathbb{Z}_N \times \mathbb{Z}_N$  which is sensitive to the projective classes of  $G_\Gamma$ . More precisely, we will be discussing subgroups  $F_\Gamma \subset G_\Gamma$  such that the homomorphism

$$\tau : H^2(G_\Gamma, U(1)) \rightarrow H^2(F_\Gamma, U(1)) \quad (5.1)$$

is bijective and use a case by case argument to show that  $F_\Gamma$  can be chosen to be of the form  $\mathbb{Z}_N \times \mathbb{Z}_N$ . Topological order can thus also be understood as “hidden” symmetry breaking if it is protected by a continuous symmetry. Note that this is not in contradiction with the Mermin-Wagner Theorem [19] (stating that spontaneous symmetry breaking does not occur in one-dimensional quantum systems with continuous symmetry) since the NL-UT only preserves the discrete subgroup  $\mathbb{Z}_N \times \mathbb{Z}_N$  but not the full continuous symmetry.

As was explained in great detail in Ref. [8], a connected compact simple Lie group can be written as a quotient  $G_\Gamma = G/\Gamma$  (which motivates the notation) where  $G$  is the universal cover of  $G_\Gamma$  and  $\Gamma$  is a subgroup of the center  $\mathcal{Z}(G)$  of  $G$ . All projective representations  $\rho$  of  $G_\Gamma$  originate from linear representations of  $G$ . The projective class of a representation of  $G_\Gamma$  can be deduced from the action of

$\Gamma$  on the corresponding linear representation of  $G$  (denoted by  $\rho : \Gamma \rightarrow U(1)$ ). With only one exception (occurring for  $G = Spin(4n)$ ),  $\Gamma$  is of the form  $\mathbb{Z}_N$ . In these cases we choose a generator  $\gamma \in \Gamma$  and define the projective class of a representation by  $\rho(\gamma) = \omega^t$  with  $\omega = \exp(2\pi i/N)$ . The main strategy in defining  $F_\Gamma$  is to start by choosing  $R, \tilde{R} \in G$  such that  $R^N \in \Gamma$  and  $\tilde{R}^N \in \Gamma$  (moreover,  $N$  should be the smallest nonzero integer for which this holds). Furthermore,  $R\tilde{R}R^{-1}\tilde{R}^{-1}$  should generate  $\Gamma$ . Thus  $R\tilde{R}R^{-1}\tilde{R}^{-1} = \gamma^m$  with  $\gcd(N, m) = 1$ . Let  $F$  be generated by  $R$  and  $\tilde{R}$ . Clearly  $\Gamma \subset F$ . Furthermore  $F/\Gamma = F_\Gamma = \mathbb{Z}_N \times \mathbb{Z}_N$  is a finite abelian group.

It is now not too hard to see that with this choice of  $F_\Gamma$  the map  $\tau$  is bijective. Let  $\rho$  be a representation of  $G$  in the projective class  $t$  ( $\rho(\gamma) = \omega^t$  for  $\gamma \in \mathcal{Z}(G)$ ). Restrict this representation to  $F$ . The projective class of this restricted representation is determined by the phase obtained upon commuting  $R$  and  $\tilde{R}$  or, in other words, by  $\rho(\gamma^m) = \omega^{tm}$ . As a consequence we find  $\tau : t \rightarrow tm$  which is bijective under the assumption that  $\gcd(N, m) = 1$ .

It remains to show that symmetry generators  $R$  and  $\tilde{R}$  with the desired properties indeed exist. In what follows, we will provide an explicit realization for  $G$  being equal to either of the groups  $SU(N)$ ,  $Sp(N)$  or  $Spin(N)$ . A summary of the possible quotient groups to be considered can be found in [8].

**Case  $G_\Gamma = SU(N)/\mathbb{Z}_N$ :** The subgroup  $\mathbb{Z}_N$  is generated by  $\omega\mathbb{I}$ , where  $\omega = \exp(2\pi i/N)$ . We will give matrix representations of  $R$  and  $\tilde{R}$  acting on  $\mathbb{C}^N$ . Let  $v_i$  denote the standard orthonormal basis of this space. Choose  $R$  to be proportional to the linear map  $v_i \rightarrow \omega^i v_i$ . The proportionality constant  $c$  should be chosen such that  $R$  has determinant 1, thus  $c^N = (-1)^{N+1}$ . Similarly, let  $\tilde{R}$  be proportional to the linear map  $v_i \rightarrow v_{i+1}$  (and  $v_N \rightarrow v_1$ ) with the same proportionality constant  $c$ . These matrices obey  $R^N = \tilde{R}^N = (-1)^{N+1}\mathbb{I}$  and  $R\tilde{R} = \omega\tilde{R}R$ .

**Case  $G_\Gamma = SU(N)/\mathbb{Z}_q$ :** The subgroup  $\mathbb{Z}_q$  is generated by  $\omega\mathbb{I}$ , where  $\omega = \exp(2\pi i/q)$ . Choose the generators of  $F$  to be the block diagonal matrices  $R = \text{diag}(R_q, \dots, R_q)$  and  $\tilde{R} = \text{diag}(\tilde{R}_q, \dots, \tilde{R}_q)$  where  $R_q$  and  $\tilde{R}_q$  are defined just as for the case  $SU(q)/\mathbb{Z}_q$  discussed before. From the previous paragraph we then directly conclude that  $R^q = \tilde{R}^q = (-1)^{q+1}\mathbb{I}$  and  $R\tilde{R} = \omega\tilde{R}R$ .

**Case  $G_\Gamma = Sp(2N)/\mathbb{Z}_2$ :** The group  $Sp(2N)$  consists of complex unitary  $2N \times 2N$  matrices  $M$  which preserve a symplectic form  $Q$  (i.e.  $M^T Q M = Q$  where  $Q$  is non-singular skew symmetric matrix). Choose a basis such that  $Q = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$ . The group  $\mathbb{Z}_2$  is  $\{\mathbb{I}, -\mathbb{I}\}$ . Let the generators of  $F$  be  $R = \begin{pmatrix} i\mathbb{I} & 0 \\ 0 & -i\mathbb{I} \end{pmatrix}$ , and  $\tilde{R} = Q$ . Clearly  $R^2 = \tilde{R}^2 = -\mathbb{I}$  and  $R\tilde{R} = -\tilde{R}R$ .

**Case  $G_\Gamma = Spin(N)/\mathbb{Z}_2 = SO(N)$ ,  $N \geq 3$ :** The group  $Spin(N)$  is most easily understood by first considering its Lie algebra  $so(N)$ . Let  $e_i$  be an orthonormal basis of  $\mathbb{R}^N$ . The Clifford algebra  $\text{Cl}(N)$  is generated by this basis together with the relations  $\{e_i, e_j\} = 2\delta_{ij}$ . The Lie algebra  $so(N) \subset \text{Cl}(N)$  is generated by the operators  $S_{ij} = \frac{i}{2}e_i e_j$  (with  $i \neq j$ ). These operators generate rotations in the  $(i, j)$ -plane:  $T_{ij}(\theta) = e^{i\theta S_{ij}} = \cos(\frac{\theta}{2})\mathbb{I} - \sin(\frac{\theta}{2})e_i e_j$ . The group  $\mathbb{Z}_2$  is  $\{\mathbb{I}, -\mathbb{I}\}$ . The generators of  $F$  can be chosen to be  $R = e_1 e_2$  and  $\tilde{R} = e_2 e_3$ . These elements square to  $-\mathbb{I}$  and anti-commute. From the perspective of  $SO(N)$ , they correspond to  $\pi$ -rotations in two orthogonal planes with a one-dimensional intersection.

**Case  $G_\Gamma = Spin(N)/\mathbb{Z}_4 = PSO(N)$ ,  $N = 4n + 2$ :** In this case the center of  $Spin(N)$  is isomorphic to  $\mathbb{Z}_4$  and is generated by the element  $\gamma = \prod_i e_i$  (indeed  $\gamma^2 = -\mathbb{I}$ ). The generators of  $F$  can be chosen to be  $R = 2^{-1/2}(1 + e_{N-1}e_N) \prod_{i=1}^n e_{4i-2}e_{4i}$  and  $\tilde{R} = 2^{-n}e_{N-2}e_N \prod_{i=1}^{2n}(1 + e_{2i-1}e_{2i})$ . One can check that  $R^4 = \tilde{R}^4 = -\mathbb{I}$  and that  $R\tilde{R} = \gamma\tilde{R}R$ .

**Case  $G_\Gamma = Spin(N)/(\mathbb{Z}_2 \times \mathbb{Z}_2) = PSO(N)$ ,  $N = 4n$ :** This is the only exception to the above recipe since the center of  $Spin(N)$  is no longer of the form  $\mathbb{Z}_q$  if  $N$  is a multiple of four. However

we can still give a flavor of how this exception should be treated. Again, let  $\gamma = \prod_i e_i$ . The center is generated by  $\gamma$  and  $-\mathbb{I}$  (indeed  $\gamma^2 = \mathbb{I}$ ). The projective class of a representation  $\rho$  is determined by both  $\rho(\gamma) = \pm \mathbb{I}$  and by  $\rho(-\mathbb{I}) = \pm \mathbb{I}$ . In this case we shall thus need two string order parameters to determine both pre-factors via the selection rule. One could define two different NL-UTs mapping these string order parameters to Landau order parameters. In order to define both the string order parameters as well as the NL-UTs, one would need to go through the above procedure twice. That is, define  $R_i, \tilde{R}_i \in Spin(N)$  for  $i = 1, 2$  such that  $R_1 \tilde{R}_1 = -\tilde{R}_1 R_1$  and  $R_2 \tilde{R}_2 = \gamma \tilde{R}_2 R_2$ . Let  $R_1 = R_2 = \prod_{i=1}^n e_{4i-2} e_{4i}$ ,  $\tilde{R}_1 = e_1 e_2$  and  $\tilde{R}_2 = 2^{-n} \prod_{i=1}^{2n} (1 + e_{2i-1} e_{2i})$ , which satisfy the desired conditions (together with  $R_1^2 = (-1)^n \mathbb{I}$ ,  $\tilde{R}_1^2 = -\mathbb{I}$  and  $\tilde{R}_2^2 = \gamma$ ).

**Case  $G_\Gamma = Spin(N)/\mathbb{Z}_2 = SS(N)$ ,  $N = 4n$ :** As discussed before, the group  $Spin(N)$  has center  $\{\mathbb{I}, -\mathbb{I}, \gamma, -\gamma\} = \mathbb{Z}_2 \times \mathbb{Z}_2$ . After dividing out  $\{\mathbb{I}, \gamma\}$  or  $\{\mathbb{I}, -\gamma\}$  (which are both isomorphic to  $\mathbb{Z}_2$ ) one obtains the group  $SS(N)$  also known as the semi-spinor group (see, e.g., Ref. [8]). In the former case one could define  $R = R_2$  and  $\tilde{R} = \tilde{R}_2$ , where  $R_2$  and  $\tilde{R}_2$  have been defined in the previous paragraph. Using the same reasoning as before, we directly obtain  $R\tilde{R} = \gamma\tilde{R}R$  and  $\tilde{R}^2 = \gamma\mathbb{I}$ . However,  $R^2 = -\mathbb{I} \notin \Gamma$ . Thus although  $F_\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_2$  constructed in this way does lead to a bijective  $\tau$ , it is not of the form  $\mathbb{Z}_r \times \mathbb{Z}_r$ . In principle the results discussed in Sect. 3 and 4 can not be directly applied. However, Eq. (3.1) can still be used to define a NL-UT and an analysis similar to Sect. 4 can be performed to find out what type of symmetry breaking phase the topological non-trivial phase is mapped to by such a NL-UT. In the case where  $\Gamma = \mathbb{Z}_2 = \{\mathbb{I}, -\gamma\}$  similar problems occur. One could still define  $R = R_2$  and  $\tilde{R} = e_1 e_2 \tilde{R}_2$  such that  $R\tilde{R} = -\gamma\tilde{R}R$  and  $\tilde{R}^2 = -\gamma\mathbb{I}$ . But also in this case  $R^2 = -\mathbb{I} \notin \Gamma$  such that  $F_\Gamma = \mathbb{Z}_4 \times \mathbb{Z}_2$ .

## 6 Conclusions and outlook

We presented a non-local unitary transformation which maps topological phases of  $\mathbb{Z}_N \times \mathbb{Z}_N$  spin chains to symmetry breaking ones. Since the map transfers non-local string order to the more familiar local Landau order it provides a useful alternative characterization of topological phases. Our result may be regarded as a two-fold generalization of the “hidden” symmetry breaking that is familiar from the AKLT model [15]. First of all, our method is able to deal with the existence of several distinct non-trivial topological phases, not just a single one. Moreover, in view of the existence of non-trivial subgroups  $\mathbb{Z}_r \times \mathbb{Z}_r \subset \mathbb{Z}_N \times \mathbb{Z}_N$ , we are also capable of characterizing phases which exhibit a mixture of topological protection and spontaneous symmetry breaking.

As pointed out in Sect. 5, our previous considerations lead to a full characterization of topological phases in spin systems with continuous symmetry groups. This observation relies on the existence of discrete subgroups of type  $\mathbb{Z}_N \times \mathbb{Z}_N$  in all classical groups of unitary, orthogonal or symplectic type. Besides constructing these subgroups explicitly, we also showed that the projective representations of the continuous groups and their subgroups are (with one exception) in one-to-one correspondence if  $N$  is chosen properly. For the stability of edge modes it is thus not important to preserve the full continuous symmetry group. Rather it is sufficient to preserve the corresponding discrete subgroup. This phenomenon has been known for some time in the case of  $SO(3)$  [17, 18] but the picture that emerges from our paper is somewhat more complete.

As a by-product, our analysis provides a complementary perspective on the hierarchy of topological phases that was pointed out in Ref. [8]. As was shown in Ref. [8], there is an injection of topological phases when viewing the same system from the perspective of either  $G_\Gamma$  or  $G_{\Gamma'}$ , with  $\Gamma' \subset \Gamma \subset \mathcal{Z}(G)$  being two central subgroups of  $G$ . In Sect. 5 we have shown that  $G_\Gamma$  has a subgroup  $F_\Gamma$  that can be used to characterize the topological phases, and a similar statement holds for  $\Gamma'$ . From the construction of  $F_\Gamma$  it is clear that  $F_{\Gamma'} \subset F_\Gamma$ . The original hierarchy of topological phases is thus also reflected on the level of “hidden” symmetry breaking.

We would like to stress that our results from Sect. 5 also provide a precise route to embed spin systems with discrete spin degrees of freedom into spin systems with continuous symmetry. While a priori the latter appear to be more constrained, this reformulation may nevertheless be useful with regard to constructing effective low energy topological field theories in terms of non-linear  $\sigma$ -models. In this sense, it may be used to make some of the ideas discussed in Ref. [20] more precise. It remains to be clarified, however, whether the embeddings just mentioned only capture the features of topological protection with regard to continuous symmetry groups in one-dimensional systems or whether the correspondence also lifts to higher dimensions.

\* \* \*    *Note*    \* \* \*

While in the process of preparing this paper, the preprint [21] appeared which has considerable overlap with our own results. Let us briefly summarize the main differences. Our setup is, in a sense, more limited. Instead of considering arbitrary Abelian groups, we restrict our attention to groups of type  $\mathbb{Z}_N \times \mathbb{Z}_N$ . Besides, the authors of Ref. [21] construct a different non-local unitary transformation for each individual purely topological phase which can be used to map it to a phase with *full* spontaneous symmetry breaking. In contrast, we keep the non-local unitary transformation fixed and investigate the fate of various phases under this map. This allows us to investigate phases which involve a mixture of topological protection and symmetry breaking. At the end of Sect. 4 we determine precise conditions under which a given phase can possibly be mapped to a fully symmetry broken one. Finally, our treatment of continuous groups exhausts *all* classical cases and does not just cover  $PSU(N)$  and  $SO(2N + 1)$  as in Ref. [21].

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